CHAPTER 10

DERIVATION OF THE LAW OF SENSATION

THE DUAL FORMS OF THE EMPIRICAL LAW OF SENSATION: FECHNER'S AND STEVENS' LAWS

We proceed, now, in the flow diagram of Figure 1.2, toward the center block: experimental evaluation of $F = kH$. This process will occupy several chapters and involves, essentially, the evaluation of the function (9.20)

$$F = \frac{1}{2} k \ln(1 + \beta I^n / t) \ . \tag{10.1}$$

In this chapter it will be demonstrated, if it is not already apparent to the reader, that Equation (10.1) can be used to derive *both* of the common forms of the "law of sensation" that were described in some detail in Chapter 3, section I. The quest to "unify" the two forms of the law of sensation – the semilog law of Weber and Fechner (WF) with the power law of Plateau, Brentano and Stevens (PBS) – has been pursued with great vigor throughout the years. Hundreds, if not thousands, of pages have been published in this endeavor. Yet the unification emerges easily from the entropy equation (9.20) / (10.1).

You will recall how the law of sensation (or the "psychophysical law") was discovered. Stimuli of constant intensity, I, of constant duration (it is to be hoped), t', were applied to a sensory receptor in an appropriate state of adaptation (e.g. unadapted). The perceptual variable, F, was measured, and the data were graphed. From the graphed data, it became apparent that F was sometimes related linearly to the log I (WF law), and that log F was sometimes related linearly to log I (PBS law). These relationships are illustrated in Figures 3.1a and 3.1b. F, the perceptual variable, is on some occasions, taken as the subjective magnitude of the stimulus (e.g. how bright the light seems to be), and, on other occasions, is taken as the impulse frequency in a sensory neuron issuing from the receptor. The conundrum of the law of sensation is that *both* these two laws and *only* these two laws seem to work.

There is a body of papers dealing with different methods for measuring the subjective magnitude: the method of categories, of magnitude estimation, and of magnitude production. One of these (categories) is found by some to favor the semilog law (WF), and another (magnitude estimation) to favor the power law. While the distinction is undoubtedly important, I defer here to my psychophysical colleagues, who are better able to define the distinction. In these pages, I shall lump together all methods of measuring subjective magnitude, and refer to them as just that: subjective magnitude.

DERIVATION OF THE LAW OF SENSATION FROM THE ENTROPY EQUATION

In order to derive the two forms of the law of sensation from Equation (10.1), we set $t = t'$ = constant, and let

$$\gamma = \beta / t' \ . \tag{10.2}$$

Thus

$$\boxed{F = \tfrac{1}{2}k\ln(1+\gamma I^n)\ .} \tag{10.3}$$

Case (i) where $\gamma I^n >> 1$

From Equation (10.3) we have

$$F \simeq \tfrac{1}{2}k\ln(\gamma I^n) \tag{10.4}$$

$$F = \tfrac{1}{2}kn\ln I + \tfrac{1}{2}k\ln\gamma\ . \tag{10.5}$$

or

$$F = a\ln I + b = a'\log I + b\ , \tag{10.5a}$$

which is Equation (3.4) stating the WF law (Norwich, 1977). Notice that we did not have to begin as Weber did by asserting (Equation (3.1)) that

$$\Delta I / I = \text{constant},$$

(which is true for a limited range of ΔI), nor by setting ΔF to be constant for a jnd, as Fechner did.

Case (ii) where $\gamma I^n << 1$

Utilizing the Taylor expansion for $\ln(1+x)$ where $0 < x \leq 1$,

$$\ln(1+x) = x - \left(\tfrac{1}{2}\right)x^2 + \left(\tfrac{1}{3}\right)x^3 - \ ... \tag{10.6}$$

we find from (10.3)

$$F \simeq \tfrac{1}{2}k\gamma I^n - \tfrac{1}{4}k\gamma^2 I^{2n} + \text{higher order terms.} \tag{10.7}$$

Retaining only the first order term,

$$F = \frac{1}{2}k\gamma I^n\ , \tag{10.8}$$

which is identical with Equation (3.7) with a different representation of the constant (Norwich, 1977). Taking logs of both sides of Equation (10.8),

$$\log F = n\log I + \log(\tfrac{1}{2}k\gamma)\ . \tag{10.9}$$

It is clear, I think, that the two forms of the law of sensation emerge as γI^n approaches each of the two extreme values. Between the extremes, one or both of the two forms will appear to be valid (see, for example, Figures 5.1 and 5.2 of Norwich, 1991). The most general law of sensation, which embraces both the WF and PBS laws, is the entropy law (10.3).

It is seen that as I becomes large so that γI^n is not << 1, the approximation of Equation (10.9) weakens. Instead of the first order approximation, (10.8), we need at least the second order approximation, (10.7). That is, due to subtraction of the term $\tfrac{1}{4}k\gamma^2 I^{2n}$, "true" F is *less* than $\tfrac{1}{2}k\gamma I^n$. Therefore, in a log-log plot of F vs. I, the data points with higher values of I fall *below* the straight line. This result was observed for Stevens' taste data, as shown in Figure 3.1 (a). This phenomenon, wherein data with larger I-values fall below the expected straight line was observed by Atkinson (1982) for many sensory modalities. Conversely, when I becomes small, so that γI^n is not >> 1, the approximation of Equation (10.4) weakens. Therefore in a graph of F vs. $\log I$, data points with lower values of I fall *above* the straight line. This result, also, was seen in Stevens' taste data, as shown in Figure 3.1 (b). However, the most important observation is that between the two extremes, by and large, both the logarithmic law and the power law (Weber-Fechner and PBS laws) provide good approximations to the data.

The mystery of the dual form of the law of sensation would seem to be solved. Moreover, since the common property of all modalities of sensation is to transmit information, we see the *reason* for a common law of sensation. Henceforth, I use Equation (10.3) as the most general and the most meaningful form of the law of sensation.

OBJECTIONS TO A COMMON LAW OF SENSATION

The opinion has often been expressed that it is unreasonable to expect a single law to govern the operation of many sensory modalities. The most cogent objection I have encountered was put forward by Weiss (1981). Weiss draws our attention to the arbitrary nature of the measuring scale that is used to measure the physical stimulus. Suppose, using one scale of measurement, the stimulus intensity is found to be I units. Suppose, moreover, that this measure of I agreed with the law of sensation,

$$F = f(I) \ . \tag{10.10}$$

However, some other investigator decides to use a different scale of measurement, so that his / her measurement of stimulus is found to be I' units, where $I' = g(I)$, and where g is some function of I. For example, it may be that $I' = \log I$. Then, in general,

$$F \neq f(I') \ . \tag{10.10a}$$

That is, the law of sensation will not be valid when intensity is measured using the latter scale of measurement. Using the above example,

$$f(\log I) \neq f(I) \ .$$

Weiss' arguments are quite correct; but his conclusion – that no universal law of sensation is possible – is too severe. The appropriate conclusion is that there must be rules set forth governing the selection of a scale of measurement for the magnitude of the physical stimulus of a given modality, if that modality is to be governed by a common law of sensation. Within the entropy theory, that rule is given by Equation (9.14a),

$$\sigma_S^2 \propto I^n :$$

the variance of the stimulus signal must vary as the n^{th} power of the physical magnitude of the stimulus. If Equation (9.14a) holds for the measure I, then, in general, it will fail to hold for $I' = g(I)$. There are, of course, transformations, g, that still enable Equation (9.14a); to wit, Weiss' example from audition. That is, if I is sound intensity, and I' is sound pressure, then g expresses the physical relation, $I' = I^{1/2}$, and, hence

$$\sigma_S^2 \propto I^n = (I'^2)^n = (I')^{2n} = (I')^m \ ,$$

in agreement with Equation (9.14a). Hence, the law of sensation can be expressed using either intensity or sound pressure units.

However, measuring distance in logarithmic units using a slide rule would violate Equation (9.14a).

OBJECTIONS TO THE ENTROPIC FORMULATION OF THE LAW OF SENSATION

Only a few objections have been voiced specifically to the entropic form of the law of sensation through the years. People often ask how one can determine *a priori* the magnitude of the quantity γI^n relative to unity, so that one might know which form of the two empirical laws will best hold. Unfortunately, we do not know, *a priori*, the value of the constant, γ, which is, itself, made up of several constituent constants. The most reasonable approach, in my opinion, is to use the general form of the law, Equation (10.3), rather than either of the two approximations.

One may wonder why nature seems to operate at the "boundary" between the two approximate forms of the entropy law, rather than at one or other of the extremes. Why, for example, does all sensation not occur in the region $\gamma I^n >> 1$, so that the semilog law (WF) would always be valid, or, conversely, in the region $\gamma I^n << 1$ (PBS)? I have no answer to this question, but I feel that it is a very important one. A simple exploration of the function $y = \ln(1 + x)$ in this critical region has been given elsewhere (Norwich, 1991, Figure 5.2).

The entropic law has also been challenged because it does not allow for the saturation of sensory effect at high values of stimulus intensity, I. That is, in reality, when I reaches an upper limit, no further increase in F (sensation or neural impulse rate) can occur. Yet no such limit appears in Equation (10.3). While this objection is valid, it is, of course, true that no such limit appears in either of the two empirical laws either.

OTHER ENDEAVORS TO UNIFY THE TWO FORMS OF THE LAW OF SENSATION

A truly incredible volume of ink has been spent in the attempt to explain the apparent "two laws" of sensation, as evidenced, for example, by Krueger's reviews of the subject. I should like to flag only three of these endeavors – those that have impressed me the most or amused me the most, as the case may be.

The first of these, is the well-known paper by D. M. MacKay (1963), in which the author postulates that a sensory receptor emits a frequency, f_1, which is a linear function of $\log(I - I_o)$, where I_o is constant. He then assumes the presence of an internal "organizer" or "effort generator," that emits a "matching" frequency, f_2, which is a linear function of $\log(F - F_o)$, where F_o is constant. He assumes, further, that an equilibrium is achieved wherein

$$f_1 = bf_2 + \text{constant}, \tag{10.11}$$

where b is a weighting factor. He proceeds to show, algebraically, that

$$F - F_o = a(I - I_o)^n, \qquad a, n \text{ constant}, \tag{10.12}$$

which is a form of the power law of sensation. MacKay, thus, involves both semilog and power laws into one unified theory.

I confess that I do like the idea of matching frequencies using an internal frequency generator, for reasons that may become clearer toward the end of this book. However, in other respects, I find MacKay's theory wanting. A large number of *ad hoc* assumptions are invoked, in order to produce a power law of sensation. Moreover, MacKay's theory does not, to my knowledge, generalize; it accounts, in a way, for the "psychophysical law" and no other law of sensation. In contrast, the entropy equations (10.1) / (10.3) will be found to give rise to a large number of the observed laws of sensation and perception.

The second study of the two empirical laws with which I was much taken is given by Resnikoff (1989), section 2.4.[1] This author shows that there are only two possibilities for the law of sensation ("psychophysical function") which

"(1) yield constant relative information gain for 1 jnd responses, and

(2) yield relative information that is invariant under changes of scale for the stimulus measure."

The entropy equation (10.3) does, of course, embody function (2), but does not yet contain constraint (1). We shall, however, build (1) into the entropy function as an additional constraint when we come to discuss the Weber fraction.

Finally, leaving laughs last, the most sensational law of sensation may be "Nimh's Law" – so named by M. H. Birnbaum (Nimh, 1976) – which, admittedly, will always fit the data more closely than any other simple mathematical law.

We shall return to theoretical considerations toward the end of the chapter, but let us proceed now to consider how the parameters of Equation (10.3), our general law of sensation, can be evaluated.

NUMERICAL EVALUATION FO THE CONSTANTS k, γ, AND n

We note that the general law, given by Equation (10.3), contains 3 parameters, k, γ, and n, that must be estimated from experimental data, while each of the component laws, WF and PBS, each have only 2 parameters. Two parameters are all that are needed to produce a straight line in a semilog plot (WF); and two parameters are all that are needed to produce a straight line on a log-log plot (PBS). The third parameter in the general law is needed, so to speak, to incorporate the slight deviation from a straight line that is observed with either of the two plots. However, the deviation from linearity is often so slight that robust numerical estimates of all three parameters are not possible. The method for parameter estimation that is usually used is the method of curve-fitting by the least squares criterion. When one attempts to curve-fit a function of 3 parameters to data that are nearly linearly arrayed, using an appropriate computer program, it is often observed that two of the parameters tend to "trade off" with each other. That is, one parameter increases its value, perhaps over several orders of magnitude, while a second parameter decreases concurrently. The result of all this variation in parameter values is to leave the sum of squares of residuals nearly constant. The sum of squares does decrease, but the fractional change is not nearly as great as is the fractional change in the values of the parameters. I refer to such parameter values as *non robust*.

It is often easy to see the reason for the trading-off behavior of non-robust parameter values. For example, if one attempts to curve-fit the general entropy equation (10.3) to experimental data that span only the region where $\gamma I^n << 1$, the computer will not fail to oblige you. It will produce a set of numerical values for k, γ, and n. However, you will probably notice that while the sum of squares of residuals decreases only slightly, the values of k and γ change dramatically, the one increasing and the other decreasing, while the value of n remains relatively stable. The reason for this behavior can be seen from Equation (10.8), which is the approximation of the general equation for small values of γI^n:

$$F = \tfrac{1}{2}k\gamma I^n = \tfrac{1}{2}\varepsilon I^n \ . \tag{10.13}$$

We see that the *product* of parameters, $k\gamma$, is regarded by the computer as a single parameter, ε. Therefore, k and γ can change *ad libitum* without changing the calculated value of F, thus leaving the sum of squares nearly constant. These ideas are illustrated in Tables 10.1 and 10.2, and in the accompanying Figure 10.1 .

The upshot of the above is that it will be difficult, indeed, to estimate distinct values for k and γ from data that relate F to I. The value of the exponent, n, however, will be robust. We shall have to appeal to other types of data to separate out k and γ. Nonetheless, for select sets of measured data, where I spans the full physiological range of perceptible values, the data may demonstrate enough deviation from linearity that 3 distinct parameter values may be found.

Table 10.1 A Set of Eight Pairs of Numbers Selected Using Only the Criterion that when Y Is Plotted against X, the Points Will Scatter, Roughly, about a Straight Line.

X	Y
1	1.5
2	2.2
3	2.5
4	3.3
5	3.7
6	4.2
7	4.8
8	5.3

Note. A function of the form given by Equation (10.3) was fitted to these simulated data. The use of such a three-parameter function is not really appropriate to fit data that lie nearly on a straight line.

Table 10.2 Excerpts from Simplex Program Output Used to Carry Out Curve-Fitting of the Simulated Data from Table 10.1.

Iteration Number	518	776	1134	1571
Sum of squares of residuals	0.1870	0.15600	0.14300	0.13600
k	21.8600	42.13000	88.90000	164.30000
γ	0.1225	0.06357	0.02959	0.01618
n	0.7575	0.70700	0.68330	0.66840
$k\gamma$	2.6780	2.67800	2.63100	2.65800

Note. The function fitted was

$$Y = (1/2)k\ln(1 + \gamma X^n) .$$

We may observe from this table that the sum of squares of residuals diminished with progressive number of iterations (of course), but the decrease was quite small over 1000 iterations (27%). Over the same 1000 iterations, the value of n changed by only about 12%. However, k and γ each changed by a *factor* of 7.57. In fact, k and γ "traded off" in value with each other, so that the product, $k\gamma$, remained nearly constant at 2.65, as required by the Taylor series (10.6). The simulated data are plotted in Figure 10.1, together with the "best" and "poorest" fitted functions (sum of squares = 0.136 and 0.187 respectively). It may be seen that, despite the considerable differences in parameter values, the fitted curves are nearly superimposed over the range of the data.

Let us consider first the data of Stevens for taste of NaCl solutions (Figure 8 of Stevens, 1969), Figures 3.1a and 3.1b. Suppose that we first use the data of Figure 3.1a and fit $\ln F$ vs. $\ln I$ to a straight line by least squares regression. Most of us have access to a scientific hand calculator that will do the job easily. From Equation (10.9), the slope of this straight line provides the value for n. Suppose, now, we use the data of Figure 3.1b, and fit F vs. $\ln I$ to a straight line. From Equation (10.5) we see that the slope of this line is equal to $\frac{1}{2}kn$ and its F-intercept is equal to $\frac{1}{2}k\ \ln\gamma$. Since we have determined the value of n from the first graph, we can obtain the value of k from the slope of the second graph. With k determined, we can calculate the value of γ from the intercept of the second graph. However, the system is "over-determined," since the intercept of the first straight line is equal to $\ln(\frac{1}{2}k\gamma)$. The value of this intercept may be checked against the already determined values of k and γ. Usually some adjustment of values is necessary to obtain a compromise position. When I assign such problems in

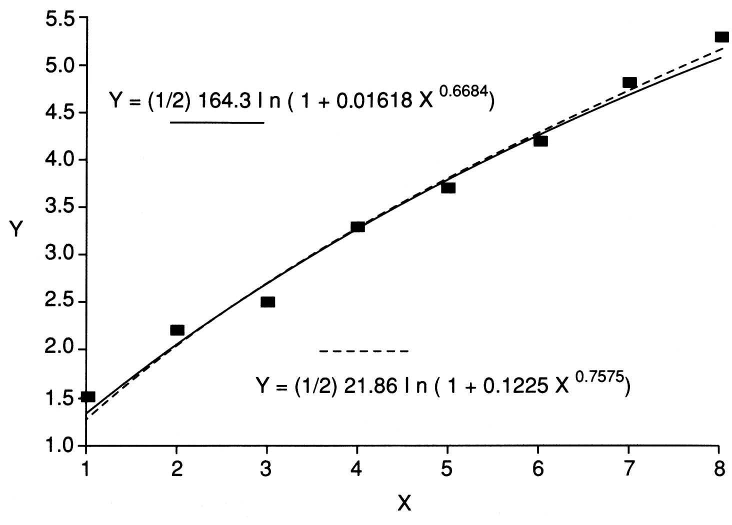

Figure 10.1 Numerical example of Table 1. Two curves fit a set of simulated data points nearly equally well, despite the fact that the parameters of the two curves differ considerably. The curves are obtained from a fit to the entropy function (10.3). This demonstrates the difficulties encountered when points that lie nearly on a straight line are fitted by a function of 3 parameters.

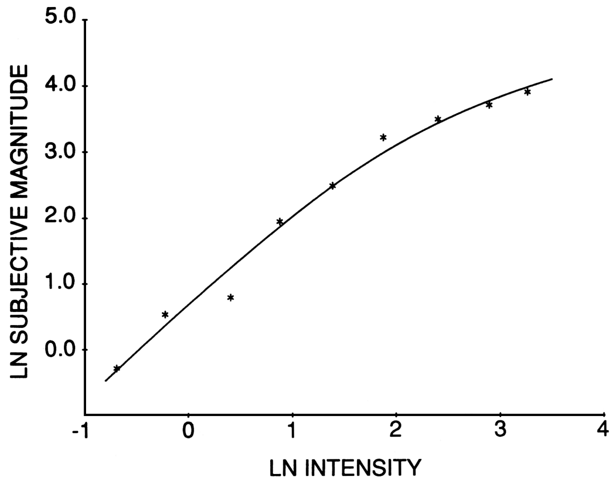

Figure 10.2 Data of S. S. Stevens (1969). Natural log of subjective magnitude of taste of sodium chloride solutions plotted against natural log of percent concentration by weight of NaCl. The entropy equation in the form of Equation (10.3) has been fitted to the data, and the resulting curve has been plotted.

$$F = (41.31/2)\ln(1 + 0.09995\, I^{1.483}) .$$

Compare with the PBS and WF laws, plotted in Figures 3.1a and b respectively. Notice how the deviation of the data points from a straight line on the full log plot is embraced naturally by the entropy equation.

curve-fitting to my students, they usually attack the problem in the manner described above, and emerge with quite reasonable and consistent values for the three parameters.

Alternatively, one can use a computer program that fits non-linear functions such as (10.3) by a process of "hill-climbing," using the least squares criterion. My own favorite is a downhill simplex routine that will provide a good fit of Equation (10.3) to the data with a few minutes' execution time on a PC. This method was devised by Nelder and Mead in 1965, but good renditions in Fortran and C computer languages can be found in Press *et al.* (1986, 1988), and a clear explanation of the algorithm together with a listing of a simplex program in Pascal is given by Caceci and Cacheris in *BYTE* magazine (1984).

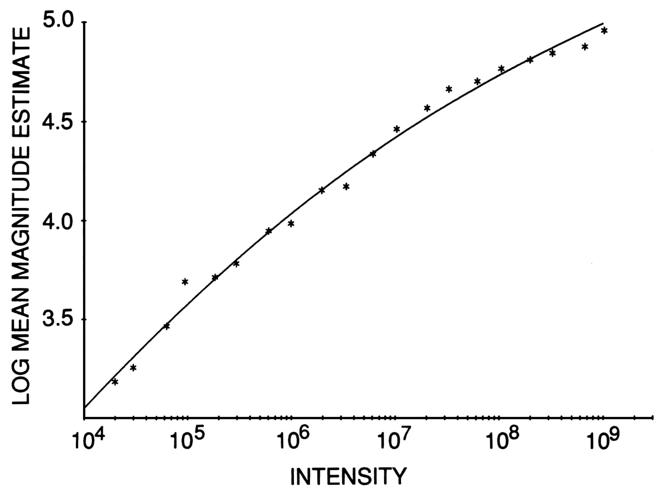

Figure 10.3 Data of Luce and Mo (1965). Natural log of mean magnitude estimate of intensity of a 1000 Hz tone (subject 9) plotted against log of sound intensity. The reader may observe in Luce and Mo's Figure 2 how the data on a log-log plot deviate characteristically from a straight line. The curvature is captured by the entropy equation:

$$F = (113.1/2)\ln(1 + 0.03131\, I^{0.2896}) .$$

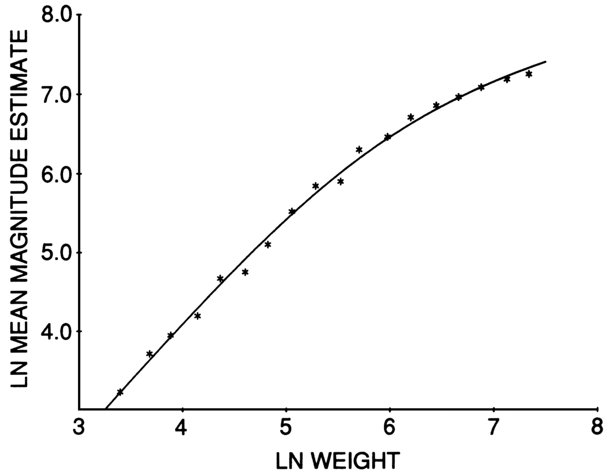

Figure 10.4 Data of Luce and Mo (1965). Natural log of mean magnitude estimate plotted against natural log of lifted weight (subject 6). The deviation of the plotted data from a straight line on a log-log plot for all 6 subjects is very clearly seen in Luce and Mo's Figure 3. The fitted entropy equation is:

$$F = (1040/2)\ln(1 + 0.0003022\, I^{1.499}) .$$

Equation (10.3) was fitted to Stevens' NaCl-taste data ($\log F$ fitted against log right-hand side of (10.3)) by the simplex method, and the result is shown in Figure 10.2. The following parameter values were obtained: $k = 41.31$, $\gamma = 0.09995$, $n = 1.483$. The value of the exponent, n, is, of course, similar to the value one would obtain from a simple regression line to a log-log plot. The value of the scaling constant, k, will take on much more significance to us after we have explored the process of adaptation.

Much of the published data relating F with I do not exhibit enough curvature in a log-log or semilog plot to permit robust estimations of k and γ. A number do, however, as in the example given above.

Figure 10.3 shows the entropy function fitted to the auditory data of Luce and Mo (1965). The following parameters were obtained by a least squares procedure: $k = 113.1$, $\gamma = 0.03131$, $n = 0.2896$. Again, the value of the exponent, n, is in accord with the value of 0.3 which is usually quoted for

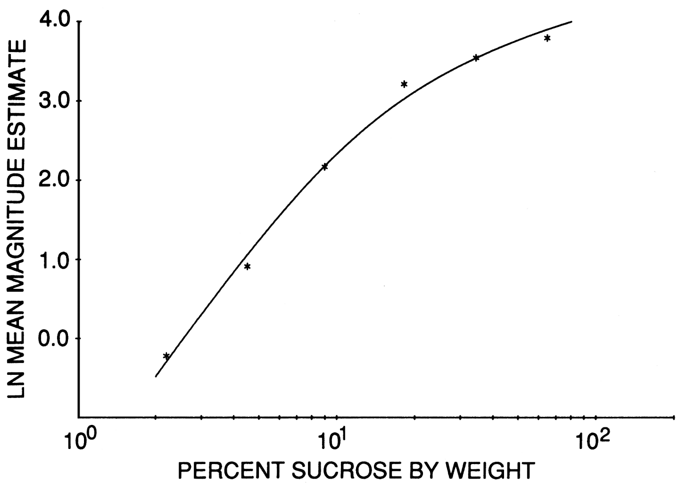

Figure 10.5 These data were digitized, approximately, from Moskowitz (1970b), Figure 1. Natural log of sweetness of sucrose is plotted against log percent sucrose by weight. The entropy function is

$$F = (24.6/2)\ln(1 + 0.0126\, I^{2.03}) .$$

The deviation of sucrose data from a straight line is seem, perhaps, even more clearly in Moskowitz's earlier paper of the same year (1970a), Figure 1.

audition. Figure 10.4 demonstrates the fit of the entropy equation to Luce and Mo's data on lifted weights. The parameter values are k =1040, $\gamma = 3.022 \times 10^{-4}$, $n = 1.499$. Data for mean magnitude estimates of sweetness of sucrose were estimated from a graph provided by Moskowitz (1970b), and the results are shown in Figure 10.5. These data are somewhat more approximate than the others. Parameters for the entropy function are: $k = 24.6$, $\gamma = 0.0126$, $n = 2.03$. The value for n is a little greater than the value that would have been obtained from the usual log-log plot.

There is magic in the values of the constant k, but to sample its enchantment you must remain apprentice to this sorcerer for at least one more chapter (or else cheat and jump ahead, but you may find yourself in deep water!).[2]

MEASUREMENTS OF TOTAL NUMBERS OF ACTION POTENTIALS

When F is interpreted as the impulse or action potential frequency in a nerve fiber, the integral $\int_0^t F dt$ gives the total number of impulses that will be recorded in the interval, t, following administration of the stimulus. Integrating Equation (10.1) with respect to t,

$$\int F\,dt = \int \tfrac{1}{2}k\ln(1 + \beta I^n/t)\,dt$$
$$= \tfrac{1}{2}kt\ln(1 + \beta I^n/t) + \tfrac{1}{2}k\beta I^n\ln(\beta I^n + t) + \text{constant.} \qquad (10.14)$$

We can now evaluate the parameters of the entropy function (10.1) by curve-fitting $\int_0^t F dt$ to data that relate total number of impulses to stimulus intensity. A recent example of this type of experiment is provided by Duchamp-Viret *et al* (1990), who measured the response in olfactory bulb neurons to the four stimuli, *DL*-camphor, anisole, *DL*-limonene and isoamyl acetate. The number of impulses in the interval 0 — 500 ms following onset of the stimulus for "all stimuli together ... pooled as a function of concentration" were plotted against concentration, in their Figure 9A. Equation (10.14) was fitted to their data, and the result is shown in Figure 10.6. Note that no further increase in total impulses per 500 ms occurs for values of $\log_{10} I$ greater than about -2. This saturation effect is not embraced by Equation (10.1), which does not recognize any physiological upper limit for the variable I. Otherwise, the curve-fit is quite good. Parameter values are $k = 59.0$, $\beta = 6.91 \times 10^4$, $n = 1.15$.

I submit here, in conclusion, a brief and quite approximate analysis of the mechanoreception data of Werner and Mountcastle (1965). I am not sure that their stimuli, which were repeated indentations of

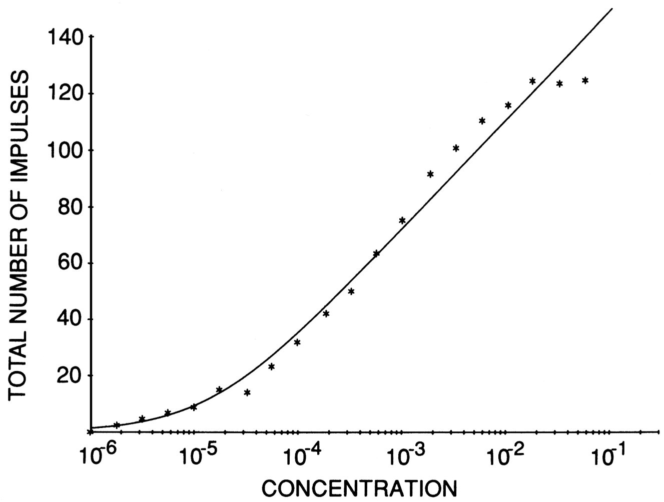

Figure 10.6 The integral of the entropy function with respect to time gives the total number of impulses expected over a given time interval. This integral, given by Equation (10.14), has been fitted to the observed total number of impulses counted in the interval 0 — 500 ms in an olfactory bulb neuron, as reported by Duchamp-Viret, Duchamp *et al.* (1990). The fitted curve (solid line) conforms reasonably well to the data, but will not show "saturation" for high values of concentration. Parameters values for the fitted curve are given in the text.

skin, qualify as simple "intensities," but the analysis offered below is easy and the results are quite striking.

A tactile probe was used to stimulate the skin of cats and monkeys. A train of 30 — 50 stimulus of strength, *I*, were delivered at intervals of 3 - 5 s. Intensity was measured in microns of skin indentation. The sensory receptors are mechanoreceptors, and impulses were counted in a single mechanoreceptive fiber. The total number of impulses were counted for a number of time intervals, such as 20, 50, 100, 250, 500 and 1000 ms. Werner and Mountcastle then plotted the logarithm of the total number of impulses counted against the logarithm of skin indentation in microns. Their result was a series of nearly-parallel straight lines (their Figure 10). To analyze these data, it is simpler to use an approximate form of the entropy equation, similar to Equation (10.8). Expand (10.1), again, in a Taylor series, retaining only the first term:

$$F = \tfrac{1}{2}k\beta I^n/t\,. \qquad (10.15)$$

Suppose that the stimulus begins at $t = 0$, and the first impulse is registered at $t = t_o$. Then we have

$$\int_{t_o}^{t} F\,d\tau = \int_{t_o}^{t} \tfrac{1}{2}k\beta I^n/\tau\,d\tau = \tfrac{1}{2}k\beta I^n \ln(t/t_o)\,. \qquad (10.16)$$

This integral is approximately equal to the total number of impulses in the time interval t_o to t. Taking logs of both sides,

$$\ln \int_{t_o}^{t} F\,d\tau = n\ln I + \ln\left[\tfrac{1}{2}k\beta\ln(t/t_o)\right]. \qquad (10.17)$$

We see that if $\ln \int_{t_o}^{t} F\,d\tau$ is plotted against ln *I* for a given, fixed value of *t*, the result expected is a straight line whose slope is equal to *n*. The intercept of this straight line is given by

$$\ln K = \ln[\tfrac{1}{2}k\beta\,\ln(t/t_o)] \;=\text{intercept.} \qquad (10.18)$$

Measured values of the quantity, *K*, have been tabulated by Werner and Mountcastle in their Table 1. For $t = t_1$, let $K = K_1$. Then, from (10.18),

$$K = (K_1 - \tfrac{1}{2}k\beta\,\ln t_1) + \frac{1}{2}k\beta\,\ln t\,. \qquad (10.19)$$

Thus, from Equation (10.18), *K* = exp (intercept of straight line) is a linear function of ln *t*, a relationship which can be tested.

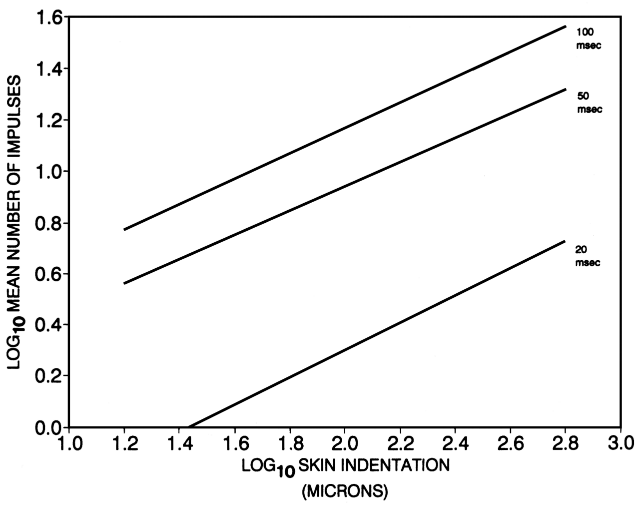

Figure 10.7a Mechanoreceptor data of Werner and Mountcastle (1965, fiber 23, Figure 10) have been represented schematically. Log_{10} of mean total number of impulses in a single mechanoreceptive fiber have been plotted against the $\log_{10}$ of stimulus (skin indentation). The result is a series of nearly-parallel straight lines. The total duration of the stimulation is indicated at the right-hand side of each line. The slope of the straight lines is, by the entropy theory, equal to the exponent, *n*, in the law of sensation.

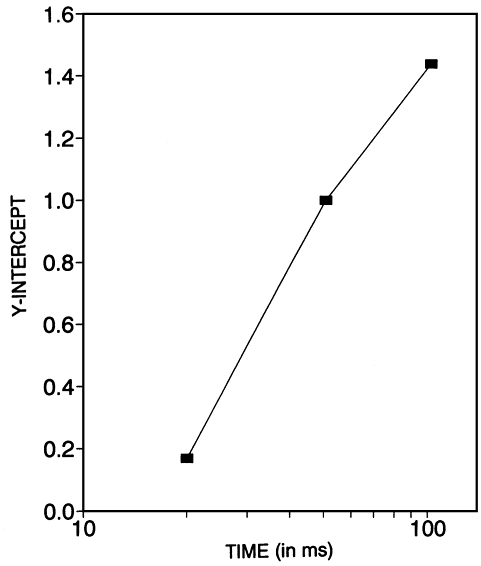

Figure 10.7b The straight lines in Figure 10.6a have the general equation

$$\log_{10} \text{mean number of impulses} = n \log_{10} \text{skin indentation} + \log_{10} K . \qquad (10.17) / (10.18)$$

In this graph, K (not $\log_{10}K$) is plotted against log t, where t is total duration of stimulation. From Equation (10.19), these 3 points are expected to lie on a straight line, which they do approximate.

Since the impulse rate in Werner and Mountcastle's study equilibrated after about 150 ms (authors' Figure 4), only summation times equal to or less than 150 ms can be used in Equation (10.16). (We could, of course, modify (10.16) to allow for equilibration.) The three straight lines for fiber 24-3 (shown also in the authors' Figure 10), corresponding to t = 20, 50, and 100 ms are drawn, approximately, in Figure (10.7a). The K-values for these straight lines (obtained from the authors' Table 1) are plotted against $\ln t$ in Figure (10.7b). We see that the three points lie nearly on a straight line, as predicted by Equation (10.18). That is, the entropy equation has shown that the data of Werner and Mountcastle in a log-log plot will lie on a series of parallel straight lines (as found experimentally by these authors), whose slope is equal to the power function exponent, and whose intercepts are proportional to $\ln(t/t_o)$. This analysis cannot be pushed too far. Equation (10.15) is only an approximation of (10.1), and no allowance has been made for the spontaneous firing rate of about 6 impulses per second.

THE PERCEPTUAL VARIABLE

The variable, F, has been termed a "perceptual variable," but what are the criteria for selecting such a variable? I have used, variously, magnitude estimates, category scales, and neural impulse rates. I suppose that I am searching for any quantity that nature appears to use as a measure of entropy, H. That definition may not be adequate, but it is the best I can offer at this time. Neither is the list exhaustive. For example, for the sense of audition, neural impulse rate may not, in itself, be an adequate measure of H. Intensity of sound is mediated more strongly by the number of nerve fibers that are firing than by the frequency of impulses in a given fiber (Coren and Ward, 1989). Perhaps, in this case, number of fibers firing is an appropriate perceptual variable.[3] For a review of the "Doctrine of Correspondence" (psychophysical to neurophysiological) the reader is referred to Marks (1978, pp 164-170). We shall begin to distiguish theoretically between F(neuronal) and F(psychophysical) in Chapter 13.

Within an evolutionary model, one might say that sensory systems evolve in a manner compatible with the equation $F = kH$. That is, nature, using the evolutionary process, refines *some* physiological mechanism (neural firing rate, number of fibers firing, even electric or magnetic field strengths) which encodes the stimulus entropy. When we have discovered the identity of F by mathematical means for some modality, we may then designate it as a "perceptual variable."

UNIFICATION

We recall Equation (3.17), where a unifying sensory function, $F = F(I, t)$, was hypothesized. This hypothesized function is now identified with the function given by Equation (10.1). The unifying function has now achieved its first goal: by setting $t = t' =$ constant, we obtain, from Equation (3.18),

$$F = F(I, t') ,$$

which is identified with Equation (10.3), which, in turn, is the unified law of sensation.

We observe that by setting $t =$ constant and defining a new constant, γ, in Equation (10.2), we have effectively removed assumption (4) (Chapter 9), the assumption of constant sampling rate. As we continue to use restricted forms of the *H*-function, obtained by setting one or another variable equal to a constant, we selectively remove a corresponding assumption from the list of six.

HISTORY

We have spoken briefly about Fechner's law in Chapter 3. This relationship between stimulus and response forms what Fechner termed "outer psychophysics." However, Fechner also wrote about "inner psychophysics," in which he conjectured that in the nervous system there exist internally generated oscillators, and that the sensation resulting from some external stimulus had to superimpose itself, in some way, upon these internal oscillations.[4] Such considerations led Delboeuf[†] (1873) to suggest a modification of Fechner's law of the form

$$F = k \log(1 + I/I_n) , \qquad (10.20)$$

where I_n is produced as a result of internal neurological activity.

Delboeuf's equation is mentioned here because of its clear resemblance to the entropy equation (10.3). One cannot but observe the similarity in Delboeuf's insertion of I_n to the required incorporation of σ_R^2 in the information equation. Where Delboeuf has used $\log(1 + I/I_n)$, we have used the information $\log(1 + \sigma_S^2/\sigma_R^2)$, which we might write as $\log(1 + I^n/I_R^n)$ [being careful about the interpretation of I_R].

We see in Chapter 12 that Bekesy (1930) utilized a similar function in his attempt to account quantitatively for the results of Knudsen on differential sensitivity, $\Delta I/I$, of audition. Bekesy attributed the derivation of the equation to Alfred Lehmann (1905).

The theoretical work of Rushton (1959) using data measured by Fuortes (1959) is also noteworthy. Working with membrane resistance in the eccentric cell in the eye of *Limulus* (crab), Rushton obtained the following equation empirically:

$$R_O - R_I = \tfrac{1}{2} \log_{10}(1 + 25\, I) , \qquad (10.21)$$

where R_I is membrane resistance in response to light intensity, I, and R_O the resistance in the dark. Rushton goes on to speculate "... further, R_I will be a linear function of impulse frequency ..." This equation also is of the same general form as the entropy Equation (10.3).

NOTES

1. Must *c* always be equal to 1 if Resnikoff's Equation (2.21) is to agree with Equation (2.20)?
2. Such was, of course, the lesson taught by Goethe.
3. Since sound intensities in the range of human hearing vary by a factor of about 10^{10}, recruitment of fibers is, in itself, unlikely to mediate loudness.
4. For a broad examination of Fechner's contributions, see the recent review by Murray (1992).

†. (2003 ed. note) The story of Delboeuf and his equation has now been published by Nicolas and Murray: Nicolas, S and Murray, D.J. The psychophysics of J-R-L Delboeuf (1831-1896), *Perception*, **26**, 1297-1315, 1997.

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