CHAPTER 15

BOLTZMANN AND BERKELEY

BOLTZMANN: $F = kH$ AND $S = -k_B H_B$

We now begin our philosophical climb up the left-hand side of the flow diagram in Figure 1.2. Although the equations are few, one requires some understanding of the ideas presented earlier in the book dealing with statistical mechanics, information theory and the foundations of the entropy theory of perception.

The H-function we have been using in our sensory studies is an H-function of the type introduced by Shannon. H (suitably constructed) can be interpreted as the uncertainty that precedes an event, or as the information gained upon perceiving the outcome of the event. However, the manner in which we have utilized the H-function is quite different from the manner in which it has been used in communications science and in psychology. As we observed in Chapter 5, information theory as used in the latter disciplines is *non-unique* in character – that is, other measures of information are not interdicted – and *extrinsic* in its application. "Extrinsic" implied that the H-measure is used as a tag or label on a given channel, which might state, for example, "This channel has transmitted 100 bits of information." It was extrinsic to the physical operation of the channel in that no equation governing the physics of the channel contained the labelled H-measure as a variable. The extrinsic property of the H-function as used in communications science and psychology is best understood in contradistinction to the intrinsic property of the H-function as used in physics. In Chapter 6, we studied, in very preliminary fashion, the use made by Boltzmann of the H-function. If H_I denotes the uncertainty of the human perceiver about which microstate is assumed by an ensemble of molecules, then Boltzmann could have written[1]

$$S = k_B H_I \, . \qquad (6.20) \,/\, (15.1)$$

That is, physical or thermodynamic entropy is equal to Boltzmann's constant (named in his honor after his death) multiplied by the H_I-function. Boltzmann's use of the H-function is both unique (no other function will do) and intrinsic. It is intrinsic because the H-measure *does* enter directly into an equation governing the physical state of matter.

The point of examining these concepts is to help put our fundamental equation,

$$F = kH \, , \qquad (2.6) \,/\, (15.2)$$

into some sort of perspective. The use of the entropy, H, in this equation is intrinsic, in the manner used by Boltzmann. The mathematical form of the H-function is unique: No mathematical function for H differing substantially from Equation (9.19) will likely possess its power to embrace all the sensory processes examined in Chapters 10 through 15. It should be absolutely clear that the intrinsic manner in which the entropy / information concept enters our equations in this entropy theory of perception differs radically from the extrinsic use of entropy / information in communications science and psychology. Our use of the information concept in the analysis of psychophysical experiments is completely different from the manner which Garner and Hake described in their well-known paper in 1951.

In order to draw perceptual theory even closer to statistical mechanics, let us look a little more deeply at the famous mathematical function known as *Boltzmann's H-function*, H_B, which was introduced in Chapter 9. Boltzmann defined this function by means of the equation

$$H_B = \int \ldots \int f \ln f \, \delta v_\mu \, , \qquad (15.3)$$

Table 15.1 Comparing H_B (Boltzmann's *H*) with *H* (Shannon's *H*), and *S* (physical entropy) with *F* (perceptual "entropy")

Entropy	*H*-function	Evolution of *H*	Evolution of entropy
$S = -k_B H_B + \text{constant}$	$H_B = \int f \ln f \, dv_\mu$	$\frac{dH}{dt} \leq 0^*$	$\frac{dS}{dt} \geq 0^\dagger$
$F = kH + \text{constant}$	$H = -\int p \ln p \, dx$	$\frac{dH}{dt} \leq 0^\ddagger$	$\frac{dF}{dt} \leq 0^\S$

Note. Discussion in main text. From the right-hand column, the evolution of physical or thermodynamic entropy in time proceeds in the opposite direction from the evolution of perceptual "entropy" (the quantity usually called the "perceptual variable"). *S* and *F* are analogous quantities.

* As a consequence of molecular collisions;
† Expression of the second law of thermodynamics;
‡ As a consequence of the central limit theorem;
§ Expression of the principle of sensory adaptation.

where δv_μ is an element in μ-space, as described in Chapter 6. The quantity, *f*, is the density of particles in μ-space; that is the number of particles per unit volume of μ-space. In fact, it can be shown (Appendix) that in an equilibrium state, apart from an additive constant,

$$H_B = -H_I \,, \tag{15.4}$$

so that, from Equation (15.1)

$$S = -k_B H_B \,. \qquad (9.26) \,/\, (15.5)$$

Let us now compare Equation (15.5) with (15.2), referring to Table 15.1. We see that in both cases, the variable on the left-hand side of the equation (*S* or *F*) represents a quantity which is measurable in the laboratory: thermodynamic entropy in one case, and, for example, neuronal impulse rate in the other. On the right-hand side, k_B, which is Boltzmann's constant, is (presumably) an immutable constant of nature, while *k* is a constant that is quite different when applied in different contexts or in different sensory systems. H_B is defined by Equation (15.3), while *H* is defined by Equation (9.23), the symbol p_{SR} being represented as *p* for simplicity. The additive "constants" come from the discussion in the Appendix in one case, and from Equation (9.25a) in the other.

Apart from minus signs, the difference between the two *H*-functions (H_B and *H*) stems from the difference between the quantities *f* and *p*. *f* is the density of particles in μ-space, while *p* is the probability density of particles in what is usually regarded as configuration space (that is, ordinary, Euclidean, 3-dimensional space). H_B is usually taken as a measure of *disorder* (Chapter 9), while *H* is related to the measure of *uncertainty*. If particles are clustered together in μ-space, so that some cells are densely packed, while others are empty, the system is regarded as being in an ordered state, and the value of H_B tends to be large. If we view μ-space as the union of configuration space with "velocity space," the ordered state represents a concentrated gas in configuration space (for example, all molecules of the gas in one corner of a room), and a region of thermal non-equilibrium in velocity space. *Boltzmann's H-theorem* showed for molecular collisions that

$$\frac{dH_B}{dt} \leq 0 \,. \qquad (9.27) \,/\, (15.6)$$

That is, H_B can never increase. Therefore, an ordered state, with larger H_B, will tend to evolve into a more disordered state with smaller H_B. Since $S = -k_B H_B$, we shall have (since $k_B > 0$) as a consequence of the minus sign,

$$\frac{dS}{dt} \geq 0 \,, \tag{15.7}$$

which states the well-known rule that physical or thermodynamic entropy tends always to increase.

Now the *H*-function, *as we have employed it* in sensory perception, also begins at its maximum value (maximum uncertainty about the mean stimulus intensity), and then, if the perceptual process continues, decreases as a consequence of the central limit theorem. That is

$$\frac{dH}{dt} \le 0 . \tag{15.8}$$

However, since $F = +kH$ (no minus sign this time),

$$\frac{dF}{dt} \le 0 . \tag{15.9}$$

So physical entropy and the perceptual variable evolve in time in opposite directions. S and F are analogous variables. They are both laboratory-determined entropies that serve as measures of their respective H-functions. These items are summarized in Table 15.1.

Letting our imagination loose, we might evoke a comparison between the evolution of the universe and the evolution of a percept. The universe, it is thought, began with a big bang in a state of low physical entropy, and is evolving ("adapting") toward a state of high physical entropy. In making a percept, we begin in a state of high informational entropy and ("evolve") adapt toward a state of low informational entropy. That is, the process of perception is a "big crunch."

GEORGE BERKELEY: THE RELATIVITY OF PERCEPTION

"... it does[2] not appear to me, that there can be any motion other than *relative*: so that to conceive motion, there must be at least conceived two bodies, whereof the distance or position in regard to each other is varied. Hence if there was one only body in being, it could not possibly be moved. This seems evident, in that the idea I have of motion does necessarily include relation."

The reader can, perhaps, identify the writer. My classes, when I read this passage, never fail to identify the writer as Albert Einstein. Einstein introduced what we now call the "special theory of relativity" in a paper entitled "On the Electrodynamics of Moving Bodies," published in 1905. The above quotation, however, is taken from Section 112, part I, of "On the Principles of Human Knowledge," by George Berkeley, published in 1710, 195 years before Einstein's famous paper [refer, for example, to the volume annotated by M.R. Ayers (1975)]. Einstein, of course, carried the idea much further, and virtually reformulated our ideas of space and time in rigorous mathematical terms. Moreover, he was probably totally unaware of Berkeley's statement (at least I can find no evidence that he had read it), and not particularly sympathetic to Berkeley's theories of perception. Is it not, however, an arguable point that Berkeley was the first modern relativist?[3]

While I hope, in the future, to explore the link between H_B and H in more detail, it was not, as I have said earlier, by analogy to $S = -k_B H_B$ that $F = kH$ was written.[4] The $F = kH$ relationship emerged in my mind from considerations of the philosophy of George Berkeley.

Berkeley (1685 – 1753) cannot properly be condensed into a portion of a chapter, which is, nonetheless, all the space we may allow him here. Born in Ireland, educated in Trinity College, Dublin, he became a Fellow of that college in 1707. Berkeley made numerous contributions to philosophy. Perhaps best known of his works are "An Essay towards a New Theory of Vision" (1709), and "Principles ..." first published the following year. It is with the latter opus that we are concerned.

Berkeley's ideas on perception cannot be appreciated fully without some understanding of the theories of John Locke, which preceded him. Locke proposed the existence of a single "substance" that constituted the essence of matter. The particles of which this substance was composed gave rise to perceivable properties such as solidity and motion, called *primary qualities*, and other properties such as color and sounds, called *secondary qualities*. Berkeley, however, denied the reality of the "substance" or substratum, that could be postulated but never directly perceived. Berkeley functionally replaced the "substance" of a body or object by a Spirit, or an active being "possessing a will."[5] If the spirit or mind were to be removed, the body or object would then cease to exist. But let us permit Berkeley to speak for himself.

"For as to what is said of the absolute existence of unthinking things without any relation to their being perceived, that seems perfectly unintelligible. Their *esse* is *percipi*, nor is it possible they should have any existence, out of the minds or thinking things which perceive them."

(Principles, Part I, Section 3)

"Some truths there are so near and obvious to the mind, that a man need only open his eyes to see them. Such I take this important one to be, to wit, that all the choir of heaven and furniture of the earth, in a word all those bodies which compose the mighty frame of the world, have not any subsistence without a mind, that their being is to be perceived or known; that consequently so long as they are not actually perceived by me, or do not exist in my mind or that of any other created spirit, they must either have no existence at all, or else subsist in the mind of some eternal spirit: it being perfectly unintelligible and involving all the absurdity of abstraction, to attribute to any single part of them an existence independent of a spirit. To be convinced of which, the reader need only reflect and try to separate in his own thoughts the being of a sensible thing from its being perceived."

(Principles, Part I, Section 6)

"It is very obvious, upon the least inquiry into our own thoughts, to know whether it be possible for us to understand what is meant, by the *absolute existence of sensible objects in themselves*, *or without the mind*. To me it is evident those words mark out either a direct contradiction, or else nothing at all. ... It is on this therefore that I insist, to wit, that the absolute existence of unthinking things are words without a meaning, or which include a contradiction." **(Principles, Part I, Section 24)**

Esse is *percipi*, wrote Berkeley. To be is to be perceived. Without a perceiver, there can be no existence. From Section 7, Part I: "Now for an idea to exist in an unperceiving thing, is a manifest contradiction; for to have an idea is all one as to perceive: that therefore wherein colour, figure, and the like qualities exist, must perceive them; hence it is clear there can be no unthinking substance or *substratum* of those ideas." So much, he thought, for Locke.

If to be is to be perceived, does the tree cease to exist when I cease to perceive it? No, replies Berkeley in Section 48. "For though we hold indeed the objects of sense to be nothing else but ideas which cannot exist unperceived; yet we may not hence conclude they have no existence except only while they are perceived by us, since there may be some other spirit that perceives them though we do not. Wherever bodies are said to have no existence without the mind, I would not be understood to mean this or that particular mind, but all minds whatsoever. It does not therefore follow from the foregoing principles, that bodies are annihilated and created every moment, or exist not at all during the intervals between our perception of them."

Does God (the "eternal spirit") play the role of Perceiver-in-Residence, retaining the universe in existence even when you and I may slumber? Berkeley is not absolutely clear on this point. He followed his Principles by a set of Three Dialogues Between Hylas and Philonous. Hylas doubts, while Philonous argues spiritedly for the principle of relative perception. It is not necessary to have read the Principles to appreciate the verbal sparring of Hylas and Philonous. In Sections 251-252 of the Dialogues, Philonous advances a theory of a kind of two-tiered existence: "When things are said to begin or end their existence, we do not mean this with regard to God, but his creatures. All objects are eternally known by God, or which is the same thing, have an eternal existence in his mind: but when things before imperceptible to creatures, are by a decree of God, made perceptible to them, then are they said to begin a relative existence, with respect to created minds."

In this book, we are concerned with the second of Philonous' existences: relative existence with respect to created minds. God does not enter our secular equations explicitly as a Variable, nor do I believe that Berkeley ever intended Him in that role.

Berkeley was, perhaps, the first person whose Weltanschauung, or world view, was completely dominated by what we in the twentieth century call the concept of relativity. He clearly foresaw the Einsteinian view of relativity of motion (the "principle of relativity"), as evidenced by Section 112; and he also comprehended in his *esse* is *percipi* "the relativity of existence."

I have never been able to refute the Berkeleian position that no meaning can be assigned to "existence" except within the mind of a perceiver. Moreover, I have always felt that as we, in the sciences, approach the core of the issue of perception, whether by the avenue of biology, psychology or physics, we shall find Berkeleian relativity awaiting us. Specifically, whether describing the effects of a light flash through psychophysics, or the space-time trajectory of the quantum of light through physics, our equations of perception will contain, immanent within them, Berkeley's relativity of existence. I began thinking about these problems as a student, in 1959, and I have carried them with me ever since.

In the course of the preceding few pages we have probed Boltzmann's equations a little deeper, and we have tried to capture the gist of Berkeley's Principles. Still, you may have no sense, yet, of how

Boltzmann's and Berkeley's ideas might merge. There is only a thin strand which might unite the two, a strand that I shall try to extend in the course of the next chapter.

APPENDIX: THE NEAR-EQUIVALENCE OF H_B AND H_I

H_I has been defined for the equilibrium state by

$$H_I = \ln W , \tag{6.19}$$

and H_B by the equation

$$H_B = \int \cdots \int f \ln f \, \delta v_\mu . \tag{15.3}$$

If we represent the integral by a summation, then

$$H_B = \sum_{i=1}^{n} f_i \ln f_i \, \delta v_\mu , \tag{A15.1}$$

where n is the total number of cells in μ-space. From the definition of f,

$$f_i = N_i / \delta v_\mu , \tag{A15.2}$$

where N_i is the number of particles in the i^{th} cell (cf. Chapter 6). Combining the latter two equations,

$$\begin{aligned} H_B &= \sum_{i=1}^{n} (N_i / \delta v_\mu) \ln(N_i / \delta v_\mu) \cdot \delta v_\mu \\ &= \sum_{i=1}^{n} N_i \ln N_i - \sum_{i=1}^{n} N_i \ln \delta v_\mu . \end{aligned}$$

Recalling that

$$\sum_{i=1}^{n} N_i = N , \tag{6.18}$$

$$H_B = \sum_{i=1}^{n} N_i \ln N_i - N \ln \delta v_\mu . \tag{A15.3}$$

Now, from Equation (6.17), the number of microstates, W, is given by

$$W = \frac{N!}{N_1! N_2! \ldots} , \tag{6.17}$$

where N is the number of phase points. Taking logarithms of both sides,

$$\ln W = \ln N! - \sum_{i=1}^{n} \ln N_i ! \tag{A15.4}$$

Using Stirling's Rule, which states that

$$\ln x! \simeq x \ln x - x , \tag{A15.5}$$

we have

$$\ln W \simeq N \ln N - N - \sum_{i=1}^{n} N_i \ln N_i + \sum_{i=1}^{n} N_i . \tag{A15.6}$$

That is, as found in Equation (6.23),

$$\ln W \simeq -\sum_{i=1}^{n} N_i \ln N_i + N \ln N . \tag{A15.7}$$

Since we see from Equations (A15.3) and (A15.7) that if n and N are constant, then

$$H_I \simeq -H_B \,, \tag{A15.8}$$

ignoring additive constants. Therefore, from Equation (6.20),

$$S = -k_B H_B \,. \tag{9.26}$$

NOTES

1. Although Boltzmann *might have* written this equation in exactly this form, he did not do so. The H_I-function, as we have defined it, was not "invented" until after his death.

2. I have change *doth* to *does* to try to disguise the century of writing.

3. I am discounting Galileo for this honor, since what is termed "the Galilean principle of relativity" does not break with our intuitive notions of space. The historians of science may berate me here.

4. Moles (1966) pp. 22-23 anticipated the $S = -k_B H_B \Leftrightarrow F = kH$ relationship when he observed the similarity between Boltzmann's $S = k_B \log W$ and Weber-Fechner's $F = k \log I$.

5. An interpretation of M. R. Ayers (1975).

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